Diffusion approximation for self-similarity of stochastic advection in Burgers' equation

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February 24, 2013

Abstract

Self-similarity of Burgers' equation with some stochastic advection is studied. In self-similar variables a stationary solution is constructed which establishes the existence of a stochastically self-similar solution for the stochastic Burgers' equation. The analysis assumes that the stochastic coefficient of advection is transformed to a white noise in the self-similar variables. Furthermore, by a diffusion approximation, the long time convergence to the self-similar solution is proved in the sense of distribution.

Keywords Self similarity; stochastic Burgers equation; diffusion approximation

1 Introduction

The deterministic Burgers' partial differential equation for a field $w(t,x)$ is

$$w_t = \nu w_{xx} - ww_x \tag{1}$$

and was proposed by Burgers [6] to help understand the statistical theory of turbulent fluid motion. Here $w(t,x)$ is analogous to the velocity field and ν represents the dissipative viscosity. To better model the randomness inherent in the presumed chaos of turbulence, the following stochastic Burgers'

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equation has been suggested [7, 8, 15, 17, 19, 33, 34] and studied recently by many people [3, 4, 9, 10, 16, 30, 31, 28]:

$$w_t = \nu w_{xx} - w w_x + h(t, x, w, w_x) \tag{2}$$

where $h(t, x, w, w_x)$ represents stochastic effects defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On bounded domains the existence and uniqueness of global solution was studied by Da Prato et al. [30] when the noise term $h(t, x, w, w_x)$ is white in time, and by Holden et al. [16] using a white noise calculus. On an unbounded domain the existence of a solution was studied by a Cole–Hopf transformation by Bertini et al. [3] with $h(t, x, w, w_x)$ an additive space-time white noise.

We consider a family of solutions with special spatial-temporal form, namely the family of *self-similar solutions*, of the stochastic Burgers' equation (2) on the unbounded real line with the particular stochastic advection $h(t, x, w, w_x) = (w\zeta)_x$ for some special space-time noise $\zeta(t, x)$ to be defined later. Here for the stochastic system, the self-similarity is in the sense of distribution which is defined later. The existence of self-similar solutions and the asymptotic emergence of self-similar solutions comprises the self-similarity of the stochastic Burgers' equation. Importantly for applications, the form of the stochastic advection, $(w\zeta)_x$, is appropriate for globally conserved quantities w. Such stochastic advection is potentially of great interest in applications as it potentially illuminates some of the stochastic nature of chaotic turbulence in fluid flows. A thorough understanding of turbulence remains an outstanding challenge and researchers are increasingly invoking stochastic terms to model its effects in important environmental applications [18, 14, 39, e.g.]. We need to know how stochastic advection affects long term dynamics.

Self-similarity is an important property of systems of physical interest, of which Burgers' equation is a special case. Many researchers have studies the existence of self-similar solutions of deterministic systems [5, 13, 24, 32, 40, e.g.], and described the asymptotic behavior of self-similar solutions [2, 13, 26, 40, e.g.]. But very little is known about self-similarity in stochastic spatio-temporal systems.

We prove the existence and emergence of self-similar solutions, in the sense of distribution, for the stochastic Burgers' equation (2) for $t \geq 1$. The stochastic advection $h = (w\zeta)_x$ for the stochastic Burgers' equation (2) transforms to a multiplicative white noise in the following self-similar variables. As in earlier research [2, 32, 37, e.g.] we introduce log-time and stretched space,

$$\tau = \log t\,, \quad \xi = \frac{x}{\sqrt{t}}\,,$$

and then define the stochastic field

$$u(\tau, \xi, \omega) = \sqrt{t} w(t, x, \omega), \quad \omega \in \Omega\,.$$

Straightforward algebra derives that in a useful sense, the SPDE (2) transforms in the similarity variables to

$$du = \left[\nu u_{\xi\xi} + \frac{1}{2}\xi u_\xi + \frac{1}{2}u - uu_\xi\right] d\tau + (u\, dW)_\xi\,. \tag{3}$$

We call a solution $w(t, x, \omega)$ to the stochastic Burgers' equation (2) a stochastically self-similar solution if the distribution of $\sqrt{t}w(t, x, \omega)$ just depends on the self-similar variable $\xi = x/\sqrt{t}$. By this definition, any stationary solution $\bar{u}(\xi, \omega)$ to equation (3) is a stochastically self-similar solution of stochastic Burgers' equation (2). In order to construct a self-similar solution of stochastic Burgers' equation (2) we assume that $W(\tau, \xi, \omega)$ is an $L^2(\mathbb{R})$ valued Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance operator Q which is detailed later.

To construct a stationary solution of the transformed SPDE (3), we consider the system in a weighted space $L^2(K)$ which is defined in the next section. First, by using energy estimates and the compact embedding results of the weighted space, we show the tightness of solutions with initial value in the space $L^2(K)$. Then the classical Bogolyubov–Krylov method [1] implies the existence of a stationary solution of the SPDE (3). Due to the multiplicative noise, the method showing the attraction of the stationary solution in the case of additive noise [37] fails here due to the appearance of the unbounded term $\dot{W}$. Instead we apply a diffusion approximation to this stationary solution. For the approximating model, which is a Burgers' type equation with a singular random perturbation (equation (46)), attraction to the stationary solutions is derived by the method used for the case of additive noise. Then by the approximation, the attraction is passed to the stationary solution of (3). Here the most difficult part is to show the effectiveness of the approximation. We follow a martingale method to show the tightness of the family of stationary solutions of the approximating model. Then passing to the limit derives the attraction of stationary solutions of the stochastic advection Burgers' equation (3).

2 Preliminary

We consider the stochastic PDE (3) in the self-similarity variables. For brevity we introduce the linear operator

$$\mathcal{L}u = \nu u_{\xi\xi} + \tfrac{1}{2}\xi u_\xi + \tfrac{1}{2}u\,.$$

Denoting the weight function by $K(\xi) = \exp\{\xi^2/4\nu\}$, introduce the following weighted functional space for exponent $p > 0$

$$L^p(K) = \left\{ u \in L^p(\mathbb{R}) : \|u\|_{L^p(K)}^p = \int_{\mathbb{R}} |u(\xi)|^p K(\xi)\, d\xi < \infty \right\},$$

and for positive integer exponent k

$$H^k(K) = \left\{ u \in L^k(K) : \|u\|_{H^k(K)}^2 = \sum_{0 \le \alpha \le k} \|D^\alpha u\|_{L^2(K)}^2 < \infty \right\}.$$

Then the linear operator $\mathcal{L}$ is self-adjoint and generates an analytic semigroup $S(\tau)$ on the space $L^2(K)$ with the domain $D(\mathcal{L}) = H^2(K)$ [20]. Further, the eigenvalues of the operator $\mathcal{L}$ are

$$\lambda_k = -\frac{k}{2}, \quad k = 0, 1, 2, \dots,$$

with the corresponding eigenfunctions $e_k(\xi)$,

$$e_0(\xi) = \frac{1}{\sqrt{4\pi\nu}} \exp\{-\xi^2/4\nu\}, \quad e_k(\xi) = c_k \partial_\xi^k e_0(\xi), \quad k = 1, 2, \dots,$$

which forms a standard orthonormal basis of $L^2(K)$ when we choose c_k as the constants such that $\|e_k\|_{L^2(K)} = 1$.

In the following we denote by $E_c = \operatorname{span}\{e_1(\xi)\}$ and

$$E_s = E_c^\perp = \left\{ u \in L^2(K) : \int_{\mathbb{R}} u(\xi)\, d\xi = 0 \right\}.$$

We also denote by P_s the linear projection from $L^2(K)$ to E_s.

The following are some important basic properties on these weighted spaces [20].

Lemma 1. *1. The embedding $H^1(K) \subset L^2(K)$ is compact.*

2. There exists $C > 0$ such that for any $u \in H^1(K)$

$$\int_{\mathbb{R}} |u(\xi)|^2 |\xi|^2 K(\xi)\, d\xi \le C \int_{\mathbb{R}} |\nabla u(\xi)|^2 K(\xi)\, d\xi.$$

3. For any $u \in H^1(K)$,

$$\frac{1}{2} \int_{\mathbb{R}} |u(\xi)|^2 K(\xi)\, d\xi \le \int_{\mathbb{R}} |\nabla u(\xi)|^2 K(\xi)\, d\xi.$$

4. *For any* $u \in E_s$,
$$\langle \mathcal{L}u, u\rangle \le -\frac{1}{2}\|u\|^2_{H^1(K)} .$$

5. *If* $u \in H^1(K)$*, then* $K^{1/2}u \in L^\infty(\mathbb{R})$.

6. *For any* $q > 2$ *and* $\epsilon > 0$ *there exists constants* $C_{\epsilon,q} > 0$ *and* $R > 0$ *, such that for any* $u \in H^1(K) \cap L^q_{loc}(\mathbb{R})$
$$\|u\|^2_{L^2(K)} \le \epsilon\|u_\xi\|^2_{L^2(K)} + C_{\epsilon,q}\|u\|^2_{L^q(B(0,R))}.$$

Remark 1. By item 3 in the above lemma, in the space $H^1(K)$ we define the norm $\|\nabla u\|_{L^2(K)}$ which is equivalent to $\|u\|_{H^1(K)}$.

Further, by the spectral properties of the linear operator $\mathcal{L}$, we define $(-\mathcal{L} + 1/2)^\gamma$ for any $\gamma \in \mathbb{R}$ [38]. Then define the Sobolev space $H^\gamma(K)$, for any $\gamma \in \mathbb{R}$, as $\mathcal{D}((-\mathcal{L} + 1/2)^{\gamma/2})$, the domain of $(-\mathcal{L} + 1/2)^{\gamma/2}$. By the embedding theorem [38], $H^{\gamma_1}(K)$ is compactly embedding into $H^{\gamma_2}(K)$ for $\gamma_1 > \gamma_2$.

We make the following assumptions on the stochastic force.

Assumption 2. 1. *The stochastic force* $t\sqrt{t}\eta = t\sqrt{t}(w\zeta)_x$ *is written, in the self-similar variables, as* $(uW)_\xi$*. Here* W *is an* $L^2(K)$*-valued Wiener process with covariance operator* Q *such that*
$$Q\varphi(\xi) = \int_{\mathbb{R}} q(\xi,\zeta)\varphi(\zeta)K(\zeta)\,d\zeta \quad \text{for any } \varphi \in L^2(K),$$
with $q(\xi,\zeta) = q(\zeta,\xi)$ *positive, and*
$$\int_{\mathbb{R}}\int_{\mathbb{R}} q(\xi,\zeta)K_\xi(\xi)K_\zeta(\zeta)\,d\xi\,d\zeta < \infty .$$
The covariance Q *shares the same eigenbasis as that of the operator* $\mathcal{L}$.

2. $W_\xi(\tau,\xi)$ *is an* $L^2(K)$*-valued Wiener process with covariance operator* Q' *such that*
$$Q'\varphi(\xi) = \int_{\mathbb{R}} q'(\xi,\zeta)\varphi(\zeta)\,d\zeta$$
with $q'(\xi,\zeta) = q'(\zeta,\xi)$ *positive, and*
$$\int_{\mathbb{R}}\int_{\mathbb{R}} q'(\xi,\zeta)K(\xi)K(\zeta)\,d\xi\,d\zeta < \infty .$$
Furthermore,
$$\operatorname{Tr} Q < \infty \quad \text{and} \quad \operatorname{Tr} Q' < \infty , \tag{4}$$
and $q(\xi) := q(\xi,\xi) \in H^2(K)$*,*
$$\|q\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \quad \text{and} \quad \|q'\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \quad \text{are small.} \tag{5}$$

From the above assumptions, $q'(\xi,\zeta) = \partial_\xi \partial_\zeta q(\xi,\zeta)$ and the Wiener process W has the series representation [29]

$$W(\tau,\xi) = \sum_{k=0}^{\infty} \sqrt{q_k} e_k(\xi) \beta_k(\tau), \tag{6}$$

where $\{\beta_k\}_k$ are independent standard Brownian motions.

Remark 2. The special assumptions on $\eta(x,t,w)$ does not exclude the existence of self-similar solutions for other cases.

For later purposes we consider the SPDE (3) on the canonical probability space $(\Omega_0, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau\in\mathbb{R}}, \mathbb{P})$ which consists of the sample path of $W(\cdot,\omega)$ in the space $C(\mathbb{R}, L^2(K))$ with Wiener measure $\mathbb{P}$ on Ω_0 [1]. To be more precise, W is the identity on Ω, with

$$\Omega_0 = \{w \in C(\mathbb{R}, L^2(K)) : w(0) = 0\}.$$

Let $\theta_\tau : (\Omega_0, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau\in\mathbb{R}}, \mathbb{P}) \to (\Omega_0, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau\in\mathbb{R}}, \mathbb{P})$ be a metric dynamical system (driven system), that is,

- $\theta_0 = \mathrm{id}$,
- $\theta_\tau \theta_s = \theta_{\tau+s}$ for all $s,\ \tau \in \mathbb{R}$,
- the map $(\tau,\omega) \mapsto \theta_\tau\omega$ is measurable and $\theta_\tau\mathbb{P} = \mathbb{P}$ for all $\tau \in \mathbb{R}$.

On this canonical probability space Ω_0, we choose θ_τ to be the Wiener shift [1]

$$\theta_\tau\omega(\cdot) = \omega(\cdot + \tau) - \omega(\tau), \quad \tau \in \mathbb{R}, \quad \omega \in \Omega_0, \tag{7}$$

which preserves the Wiener measure $\mathbb{P}$ on Ω_0. Furthermore, θ_τ is ergodic under Wiener measure $\mathbb{P}$. Writing $W(\tau,\xi)$ as $W(\tau,\xi,\omega)$ to explicitly show the dependence on $\omega \in \Omega_0$, then

$$W(\cdot,\xi,\theta_\tau\omega) = W(\cdot + \tau,\xi,\omega) - W(\tau,\xi,\omega).$$

Remark 3. Notice that Ω_0 is different from Ω. However, by the self similarity transformation, for any $\omega \in \Omega$, there is a sample path of $W(\tau,\xi)$, then there exists an $\omega_0 \in \Omega_0$ that represents the sample path of $W(\tau,\xi)$ in Ω_0. The reverse is same. So in effect the space Ω_0 is the same as Ω.

Recall that a random process $\{u(\tau)\}_{\tau\ge 0}$ is said to be stationary if its joint probability distribution does not change when shifted in time τ [1]. For the SPDE (3), to construct a stationary solution it is convenient to consider the transition semigroup associated to equation (3). We define $\{P_\tau\}_{\tau\ge 0}$ on the

space consisting of bounded continuous functions $\psi : L^2(K) \cap L^\infty(\mathbb{R}) \to \mathbb{R}$ by [29]

$$(P_\tau \psi)(u^0) = \mathbb{E}\psi(u(\tau; u^0)), \tag{8}$$

where $u(\tau; u^0)$ is the solution of equation (3) with initial value $u^0 \in L^2(K) \cap L^\infty(\mathbb{R})$. Denote by $\mathcal{M}$ the space consisting all probability measures on the space $L^2(K) \cap L^\infty(\mathbb{R})$ and endow $\mathcal{M}$ with the topology of weak convergence. Define the dual semigroup $\{P_\tau^*\}_{\tau \geq 0}$ acting on $\mathcal{M}$ as

$$\int_{L^2(K)\cap L^\infty(\mathbb{R})} \psi(u)(P_\tau^* \mu)(du) = \int_{L^2(K)\cap L^\infty(\mathbb{R})} (P_\tau \psi)(u)\mu(du)$$

for any $\mu \in \mathcal{M}$ and bounded continuous function $\psi : L^2(K) \cap L^\infty(\mathbb{R}) \to \mathbb{R}$. If $\mathcal{L}(u^0)$, the distribution of initial values u^0, equals μ, then $P_\tau^* \mu$ is the distribution of the solution $u(\tau; u_0)$ [29, Proposition 11.1]. Sometimes $\mathcal{M}$ is too large, so we need the smaller space

$$\mathcal{M}_2 = \left\{ \mu \in \mathcal{M} : \int_{L^2(K)\cap L^\infty(\mathbb{R})} \|u\|^2_{L^2(K)} \mu(du) < \infty \right\}.$$

A probability space $\mu \in \mathcal{M}$ is said to be a stationary measure for the stochastic Burgers' equation (3) if

$$P_\tau^* \mu = \mu\,, \quad \text{for all} \quad t > 0\,.$$

The following property of stationary measure is useful [29, Proposition 11.5].
Lemma 3. *If $\mu \in \mathcal{M}$ is a stationary measure for (3) and the initial value u^0 is $\mathcal{F}_0$ measurable with $\mathcal{L}(u^0) = \mu$, then the solution process $\bar{u}(\tau; u^0)$ is a stationary solution to the stochastic Burgers' equation (3).*

3 Existence of self-similar solutions

By definition, a stationary solution to the SPDE (3) is a stochastically self-similar solution to the stochastic Burgers' equation (2). Next we construct a stationary solution to the SPDE (3) from any initial value $u_0 \in L^2(K) \cap L^\infty(\mathbb{R})$.

For any $\tau > 0$, in the mild sense, the transformed stochastic Burgers' SPDE (3) is written as

$$u(\tau) = S(\tau)u_0 + \int_0^\tau S(\tau - s)u(s)u_\xi(s)\,ds + \int_0^\tau S(\tau - s)(u(s)dW(s))_\xi\,. \tag{9}$$

Then by the standard method for the existence of mild solutions to SPDEs [29] we obtain the following theorem.

Theorem 4. *Assume Assumption 2 holds. For any $T > 0$ and initial value $u_0 \in L^2(K) \cap L^\infty(\mathbb{R})$, there is a unique solution $u(\tau, \xi)$ to* SPDE *(2) in $L^2(\Omega, C(0,T; L^2(K)) \cap L^2(0,T; H^1(K)))$.*

We construct a stationary solution by the Bogolyubov–Krylov method. For this we prove the tightness of the distribution of $u(\tau, \xi)$, $\{\mathcal{L}(u(\tau, \xi))\}_{\tau>0}$, in the space $L^2(K)$. We need some estimates in the spaces $L^\infty(\mathbb{R})$ and $L^2(K)$.

3.1 $L^\infty(\mathbb{R})$ estimates

We follow the approach for a scalar convection-diffusion equation [40] which was also applied to characterise solutions to a stochastic Burgers' equation with additive noise [37].

We introduce

$$\operatorname{sgn}(u)^+ = \begin{cases} 1, & u > 0\,, \\ 0, & u \le 0\,; \end{cases} \quad \text{and} \quad \operatorname{sgn}(u)^- = \begin{cases} 1, & u < 0\,, \\ 0, & u \ge 0\,. \end{cases}$$

Then for $u \in L^2(\mathbb{R})$ with $\nabla u(t) \in L^2(\mathbb{R})$, the integral $\int_{\mathbb{R}} u_{\xi\xi}\phi(u)\, d\xi \le 0$ (≥ 0) for any nondecreasing (nonincreasing) $\phi \in C^1(\mathbb{R})$. By a density discussion the integral $\int_{\mathbb{R}} u_{\xi\xi}\operatorname{sgn}(u)^+\, d\xi \le 0$ (≥ 0). Moreover, the integrals $\int_{\mathbb{R}} uu_\xi \operatorname{sgn}(u)^\pm\, d\xi = 0$ and $\int_{\mathbb{R}}(\xi u_\xi + u)\operatorname{sgn}(u)^\pm\, d\xi = 0$. Denote by $u^\pm = \operatorname{sgn}(u)^\pm u$. Let $m = \|u_0\|_{L^\infty(\mathbb{R})}$. Then multiplying $\operatorname{sgn}(u-m)^+$ and $\operatorname{sgn}(u^\epsilon - m)^+$ on both sides of (3) and integrating on $\mathbb{R} \times [0, \tau]$ with $\tau > 0$, the integral

$$\int_{\mathbb{R}} (u(\tau, \xi) - m)^+\, d\xi \le 0\,.$$

Therefore, $u(\tau, \xi) \le m$ for any $\tau > 0$. Similarly $u(\tau, \xi) \ge -m$ for $\tau > 0$. Then $\|u(\tau)\|_{L^\infty(\mathbb{R})} \le m$ for all $\tau > 0$.

3.2 Estimates in the space $H^1(K)$

We first give a uniform estimate in the space $L^2(K)$.

Let $u(\tau, \xi) = u_c(\tau, \xi) + u_s(\tau, \xi)$ with $u_c \in E_c$ and $u_s \in E_s$. Then

$$\begin{aligned} du_c &= 0\,, \\ du_s &= [\mathcal{L}u_s - P_s(uu_\xi)]\, d\tau + P_s(udW)_\xi\,. \end{aligned}$$

So

$$u_c(\tau, \xi) = u_c(0, \xi) = \langle u_0, e_0 \rangle e_0(\xi) = \int_{\mathbb{R}} u_0(\xi)\, d\xi e_0(\xi)$$

which is totally determined by the mass of tha initial value, namely $M := \int_{\mathbb{R}} u_0(\xi)\, d\xi$.

Now applying Itô's formula to $\|u_s(\tau)\|^2_{L^2(K)}$, we obtain

$$\begin{aligned}\tfrac{1}{2}\frac{d}{d\tau}\|u_s(\tau)\|^2_{L^2(K)} &\le -\tfrac{1}{2}\|u_s\|^2_{H^1(K)} - \langle uu_\xi, u\rangle \\ &\quad + \tfrac{1}{2}\left[\|u_\xi\|^2_{\mathcal{L}_2^Q} + \|u\|^2_{\mathcal{L}_2^{Q'}}\right] + \langle (u\dot{W})_\xi, u\rangle\,.\end{aligned}$$

Here $\|\cdot\|_{\mathcal{L}_2^Q}$ and $\|\cdot\|_{\mathcal{L}_2^{Q'}}$ are the norms defined on the Hilbert–Schmidt spaces $\mathcal{L}_2(Q^{1/2}L^2(K), L^2(K))$ and $\mathcal{L}_2(Q'^{1/2}L^2(K), L^2(K))$ respectively [29]. Noticing that $\|q\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$ and $\|q'\|_{L^\infty(\mathbb{R}\times\mathbb{R})}$ are small enough (5), we have some positive constant c such that

$$\tfrac{1}{2}\frac{d}{d\tau}\|u_s(\tau)\|^2_{L^2(K)} \le -c\|u_s\|^2_{H^1(K)} + c\|u_c\|^2_{H^1(K)} - \langle uu_\xi, u\rangle + \langle (u\dot{W})_\xi, u\rangle\,.$$

Integrating by parts yields

$$\langle uu_\xi, u\rangle = -\tfrac{1}{3}\int_{\mathbb{R}} u^3 K_\xi\, d\xi\,.$$

By property 6 in Lemma 1, for any $\varepsilon, \varepsilon' > 0$ and $q > 2$, there exist positive constants C_ε, $C_{\varepsilon',q}$ and R such that

$$\begin{aligned}\left|\int_{\mathbb{R}}(u)^3K_\xi\, d\xi\right| &= \tfrac{1}{2}\left|\int_{\mathbb{R}} u\xi K^{1/2}(u)^2K^{1/2}\, d\xi\right| \\ &\le \tfrac{1}{2}\left[\int_{\mathbb{R}}(u)^2\xi^2K\, d\xi\right]^{1/2}\left[\int_{\mathbb{R}}(u)^4K\, d\xi\right]^{1/2} \\ &\le C\|u_\xi\|_{L^2(K)}\|u\|_{L^2(K)}\|u\|_{L^\infty(\mathbb{R})} \\ &\le 3\varepsilon C\|u_\xi\|^2_{L^2(K)} + 3C_\varepsilon\left[\varepsilon'\|u_\xi\|^2_{L^2(K)} + C_{\varepsilon',q}\|u\|^2_{L^q(B(0,R))}\right]\|u\|^2_{L^\infty(\mathbb{R})} \\ &\le 3\left[\varepsilon C + \varepsilon' C_\varepsilon\|u\|^2_{L^\infty(\mathbb{R})}\right]\|u_\xi\|^2_{L^2(K)} + 3C_\varepsilon C_{\varepsilon',q,R}\|u\|^4_{L^\infty(\mathbb{R})}\end{aligned}$$

with some positive constant $C_{\varepsilon',q,R}$. Then

$$|\langle uu_\xi, u\rangle| \le \left[\varepsilon C + \varepsilon' C_\varepsilon\|u\|^2_{L^\infty(\mathbb{R})}\right]\|u_\xi\|^2_{L^2(K)} + C_\varepsilon C_{\varepsilon',q,R}\|u\|^4_{L^\infty(\mathbb{R})}\,. \tag{10}$$

Then for any ε and $\varepsilon' > 0$, there are positive constants that we still denote by C_ε and $C_{\varepsilon',q,R}$ for some positive q and R such that

$$\begin{aligned}&\tfrac{1}{2}\frac{d}{d\tau}\|u_s(\tau)\|^2_{L^2(K)} \\ &\le -c\|u_s\|^2_{H^1(K)} + c\|u_c\|_{H^1(K)} + \left[\varepsilon C + \varepsilon' C_\varepsilon\|u\|^2_{L^\infty(\mathbb{R})}\right]\|u_\xi\|^2_{L^2(K)} \\ &\quad + C_\varepsilon C_{\varepsilon',q,R}\|u\|^4_{L^\infty(\mathbb{R})} + \langle (u\dot{W})_\xi, u\rangle \\ &\le \left[-c + \varepsilon C + \varepsilon' C_\varepsilon\|u\|_{L^\infty(\mathbb{R})}\right]\|u_s\|^2_{H^1(K)} \\ &\quad + \left[c + \varepsilon C + \varepsilon' C_\varepsilon\|u\|_{L^\infty(\mathbb{R})}\right]\|u_c\|^2_{H^1(K)} \\ &\quad + C_\varepsilon C_{\varepsilon',q,R}\|u\|^4_{L^\infty(\mathbb{R})} + \langle (u\dot{W})_\xi, u\rangle\,.\end{aligned}$$

Now choose ε and $\varepsilon' > 0$ small enough and since $u_c = Me_0(\xi)$, $\|u_c\|_{H^1(K)} \leq CM$ for some $C > 0$, then by the Gronwall lemma

$$\mathbb{E}\|u(\tau)\|^2_{L^2(K)\cap L^\infty(\mathbb{R})} \leq R_1\,, \quad \text{for all } \tau \geq 0\,,$$

with some positive random variable R_1.

Given any $\tau_1 > 0$, in the mild sense

$$\begin{aligned} u(\tau+\tau_1) &= S(\tau)u(\tau_1) + \int_0^\tau S(\tau-\sigma)u(\tau_1+\sigma)u_\xi(\tau_1+\sigma)\,d\sigma \\ &\quad + \int_0^\tau S(\tau-\sigma)(u(\tau_1+\sigma)dW(\tau_1+\sigma))_\xi\,. \end{aligned}$$

Then taking $H^1(K)$ norm in the above equation, for any $1 > \tau > 0$, by the properties of the semigroup $S(\tau)$ there is some positive constant C, which also depends on q and q', such that

$$\begin{aligned} &\mathbb{E}\|u(\tau+\tau_1)\|^2_{H^1(K)} \\ \leq\ & C\left(1+\frac{1}{\sqrt{\tau}}\right)^2 \mathbb{E}\|u(\tau_1)\|^2_{L^2(K)} \\ &+ C\int_0^\tau \left(1+\frac{1}{\sqrt{\tau-\sigma}}\right)^2 e^{-(\tau-\sigma)}R_1\mathbb{E}\|u(\tau_1+\sigma)\|^2_{H^1(K)}\,d\sigma \\ &+ C\int_0^\tau \left(1+\frac{1}{\sqrt{\tau-\sigma}}\right)^2 e^{-(\tau-\sigma)}\mathbb{E}\|u(\tau_1+\sigma)\|^2_{H^1(K)}\,d\sigma\,. \end{aligned}$$

By the Gronwall lemma, there is a constant $R_2 > 0$ such that

$$\mathbb{E}\|u(\tau+\tau_1)\|^2_{H^1(K)} \leq CR_2(1+\tfrac{1}{\sqrt{\tau}})^2\mathbb{E}\|u(\tau_1)\|^2_{L^2(K)}$$

for any $\tau_1 > 0$ and $0 \leq \tau \leq 1$. Then we have uniform $H^1(K)$-estimates for $u^\epsilon(\tau)$ for all $\tau > 0$. By the compact embedding $H^1(K) \subset L^2(K)$ we establish the tightness of $\{\mathcal{L}(u(\tau))\}_{\tau\geq 0}$, the distribution of $\{u(\tau)\}_\tau$, in the space $L^2(K)$. Then the classical Bogolyubov–Krylov method [1] yields the existence of one stationary measure denoted by μ. Let $\bar{u}(\tau,\xi)$ be the solution with initial value distributing as μ, then $\bar{u}(\tau,\xi)$ is a stationary solution to the transformed SPDE (3) with distribution μ (Lemma 3), and by the construction of the stationary solution, we have $\bar{u} \in H^1(K)$.

Theorem 5. *Assume Assumption 2 holds. For any initial $u_0 \in L^2(K) \cap L^\infty(\mathbb{R})$, there is a stationary solution, denoted by $\bar{u}$, such that $\bar{u} \in H^1(K)$. Further, there is a sequence τ_n, with $\tau_n \to \infty$ as $n \to \infty$, such that*

$$u(\tau_n) \text{ converges in distribution to } \bar{u} \text{ in } L^2(K)$$

as $n \to \infty$. Here $u(\tau)$ is the solution to the SPDE *(3) with initial value u_0.*

Next we show that the above convergence holds for any sequence τ_n with $\tau_n \to \infty$ as $n \to \infty$. It is impractical to follow the approach used for the stochastic Burgers' equation with additive noise [37]. As Section 1 states, we consider in the next section a Burgers' equation with random fast fluctuating advection which is an approximation to the SPDE (3).

4 Burgers' equation with random fast fluctuations: self-similar solution and stability

We consider the following randomly fluctuating advection in a Burgers' type equation

$$u^\epsilon_\tau = \mathcal{L}u^\epsilon - u^\epsilon u^\epsilon_\xi + \tfrac{1}{\sqrt{\epsilon}}(u^\epsilon \bar\eta^\epsilon)_\xi\,, \quad u^\epsilon(0) = u_0\,, \tag{11}$$

where $\bar\eta^\epsilon(\tau,\xi) = \bar\eta(\tau/\epsilon,\xi)$ in which $\bar\eta(\tau,\xi)$ is the stationary Ornstein–Uhlenbeck process solving

$$d\eta = -\eta\, d\tau + dW(\tau,\xi). \tag{12}$$

Then

$$\mathbb{E}\bar\eta^\epsilon(\tau,\xi)\bar\eta^\epsilon(s,\zeta) = \tfrac{1}{2}q(\xi,\zeta)\exp\left(-\frac{|\tau-s|}{\epsilon}\right) \tag{13}$$

and $\eta^\epsilon(\tau) \in H^1(K)$ for any $\tau > 0$. By the assumption on $W_\xi(\tau,\xi)$ we have that the process $\bar\eta^\epsilon_\xi(\tau,\xi) = \bar\eta_\xi(\tau/\epsilon,\xi)$ which solves

$$d\eta_\xi = -\eta_\xi\, dt + dW_\xi(\tau,\xi) \tag{14}$$

and also $\bar\eta^\epsilon_\xi(\tau) \in H^1(K)$ for any $\tau > 0$ with

$$\mathbb{E}\bar\eta^\epsilon_\xi(\tau,\xi)\bar\eta^\epsilon_\xi(s,\zeta) = \tfrac{1}{2}q'(\xi,\zeta)\exp\left(-\frac{|\tau-s|}{\epsilon}\right). \tag{15}$$

In the following we sometimes write ω explicitly to show the dependence on $\omega \in \Omega_0$. By the definition of θ_τ, the stationary process $\bar\eta(\tau) = \bar\eta(\tau,\omega)$ can be written as $\bar\eta(\theta_\tau\omega) := \bar\eta(0,\theta_\tau\omega)$.

We study the dynamics of the random differential equation (RDE) (11) for fixed $\epsilon > 0$. First, for any $\tau > 0$, in the mild sense the RDE (11) is written as

$$u^\epsilon(\tau) = S(\tau)u_0 + \int_0^\tau S(\tau-s)u^\epsilon(s)u^\epsilon_\xi(s)\,ds + \frac{1}{\sqrt{\epsilon}}\int_0^\tau S(\tau-s)(u^\epsilon(s)\bar\eta^\epsilon(s))_\xi\,ds\,. \tag{16}$$

Then by the classical theory for the deterministic Burgers' equation [35] this theorem follows.

Theorem 6. *Assume Assumption 2 holds. For any $T>0$ and initial $u_0 \in L^2(K)\cap L^\infty(\mathbb{R})$ the* RDE *(11) has a unique mild solution*

$$u^\epsilon \in L^2(\Omega, L^2(0,T;H^1(K))\cap C(0,T;L^2(K))).$$

Notice that the RDE (11) is an evolutionary equation with a stationary force. So the tightness of its solution immediately yields the existence of a stationary solution. We next construct a stationary solution for the RDE (11), which is attractive for any $\epsilon>0$. We follow the discussion in section 3.

First, by the same discussion as in section 3.1,

$$\|u^\epsilon(\tau)\|_{L^\infty(\mathbb{R})} \le m \quad \text{for all } \tau>0\,,$$

with $m=\|u_0\|_{L^\infty(\mathbb{R})}$. Next we give a uniform estimate in τ in the space $H^1(K)$ from the estimate in space $L^2(K)$ for u^ϵ with any fixed $\epsilon>0$. We follow the same discussion as in section 3.2.

Let $u^\epsilon(\tau,\xi)=u^\epsilon_c(\tau,\xi)+u^\epsilon_s(\tau,\xi)$ with $u_c\in E_c$ and $u_s\in E_s$. Then

$$\begin{aligned} du^\epsilon_c &= 0\,, &(17)\\ du^\epsilon_s &= \left[\mathcal{L}u^\epsilon_s - P_s(u^\epsilon u^\epsilon_\xi)\right]d\tau + \tfrac{1}{\sqrt{\epsilon}}(u^\epsilon\bar\eta^\epsilon)_\xi\,. &(18)\end{aligned}$$

So

$$u^\epsilon_c(\tau,\xi)=u^\epsilon_c(0,\xi)=\langle u_0,e_0\rangle e_0(\xi)=\int_{\mathbb{R}} u_0(\xi)\,d\xi e_0(\xi)$$

which is totally determined by the mass of the initial value $M:=\int_{\mathbb{R}}u_0(\xi)\,d\xi$.

Now multiplying u^ϵ in the space $L^2(K)$ on both sides of the RDE (11) we have

$$\tfrac12\frac{d}{d\tau}\|u^\epsilon_s\|^2_{L^2(K)} \le -\tfrac12\|u^\epsilon_s\|^2_{H^1(K)} - \langle u^\epsilon u^\epsilon_\xi, u^\epsilon\rangle + \tfrac{1}{\sqrt{\epsilon}}\langle (u^\epsilon\bar\eta^\epsilon)_\xi, u^\epsilon\rangle\,.$$

Integrating by parts,

$$\langle u^\epsilon u^\epsilon_\xi, u^\epsilon\rangle = -\tfrac13\int_{\mathbb{R}}(u^\epsilon)^3K_\xi\,d\xi\,.$$

Then by the same discussion as for (10) we have that for any $\varepsilon,\varepsilon'>0$ and $q>2$, there exist positive constants C_ε, R and $C_{\varepsilon',q,R}$ such that

$$|\langle u^\epsilon u^\epsilon_\xi, u^\epsilon\rangle| \le \left[\varepsilon C+\varepsilon' C_\varepsilon\|u^\epsilon\|^2_{L^\infty(\mathbb{R})}\right]\|u^\epsilon_\xi\|^2_{L^2(K)} + C_\varepsilon C_{\varepsilon',q,R}\|u^\epsilon\|^4_{L^\infty(\mathbb{R})}\,.$$

Moreover, for any $\varepsilon>0$ there is some positive constant C_ε such that

$$|\langle (u^\epsilon\bar\eta^\epsilon)_\xi, u^\epsilon\rangle| = \left|\langle u^\epsilon\bar\eta^\epsilon, u^\epsilon_\xi\rangle\right| \le C_\varepsilon\|\bar\eta^\epsilon\|^2_{L^2(K)}\|u^\epsilon\|^2_{L^\infty(\mathbb{R})} + \varepsilon\|u^\epsilon_\xi\|^2_{L^2(K)}\,.$$

Then for any ε and $\varepsilon' > 0$, there are positive constants that we still denote by C_ε and $C_{\varepsilon',q,R}$ for some positive q and R such that

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{d\tau}\|u_s^\epsilon(\tau)\|_{L^2(K)}^2 \\
\leq\ & -\tfrac{1}{2}\|u_s^\epsilon\|_{H^1(K)}^2 + \left[\varepsilon C + \varepsilon' C_\varepsilon \|u^\epsilon\|_{L^\infty(\mathbb{R})}^2\right] \|u_\xi^\epsilon\|_{L^2(K)}^2 \\
& + C_\varepsilon C_{\varepsilon',q,R}\|u^\epsilon\|_{L^\infty(\mathbb{R})}^4 \\
& + \tfrac{1}{\sqrt{\epsilon}}\left[C_\varepsilon \|\bar\eta^\epsilon\|_{L^2(K)}^2 \|u^\epsilon\|_{L^\infty(\mathbb{R})}^2 + \varepsilon \|u_\xi^\epsilon\|_{L^2(K)}^2\right] \\
\leq\ & \left[-\tfrac{1}{2} + \varepsilon C + \varepsilon' C_\varepsilon \|u^\epsilon\|_{L^\infty(\mathbb{R})} + \tfrac{\varepsilon}{\sqrt{\epsilon}}\right] \|u_s^\epsilon\|_{H^1(K)}^2 \\
& + \left[\varepsilon C + \varepsilon' C_\varepsilon \|u^\epsilon\|_{L^\infty(\mathbb{R})} + \tfrac{\varepsilon}{\sqrt{\epsilon}}\right] \|u_c^\epsilon\|_{H^1(K)}^2 \\
& + C_\varepsilon C_{\varepsilon',q,R}\|u^\epsilon\|_{L^\infty(\mathbb{R})}^4 + \tfrac{1}{\sqrt{\epsilon}} C_\varepsilon \|\bar\eta^\epsilon\|_{L^2(K)}^2 \|u^\epsilon\|_{L^\infty(\mathbb{R})}^2 .
\end{aligned}$$

Now choose ε and $\varepsilon' > 0$ small enough and since $u_c^\epsilon = Me_0(\xi)$, $\|u_c^\epsilon\|_{H^1(K)} \leq CM$ for some $C > 0$. Then by the Gronwall lemma

$$\|u^\epsilon(\tau,\omega)\|_{L^2(K)\cap L^\infty(\mathbb{R})} \leq R_1^\epsilon(\theta_\tau\omega), \quad \text{for all} \quad \tau \geq 0\,,$$

with some positive random variable R_1^ϵ which is tempered by the properties of $\bar\eta^\epsilon$.

Now given any $\tau_1 > 0$, in the mild sense

$$\begin{aligned}
u^\epsilon(\tau+\tau_1) \;=\;& S(\tau)u^\epsilon(\tau_1) + \int_0^\tau S(\tau-\sigma)u^\epsilon(\tau_1+\sigma)u_\xi^\epsilon(\tau_1+\sigma)\,d\sigma \\
& + \tfrac{1}{\sqrt{\epsilon}}\int_0^\tau S(\tau-\sigma)(u^\epsilon(\tau_1+\sigma)\bar\eta^\epsilon(\tau_1+\sigma))_\xi\,d\sigma\,.
\end{aligned}$$

Then, taking the $H^1(K)$ norm in the above equation, by the growth of the semigroup $S(\tau)$ for some positive constant C_τ,

$$\begin{aligned}
&\|u^\epsilon(\tau+\tau_1)\|_{H^1(K)} \\
\leq\ & C\left(1+\frac{1}{\sqrt{\tau}}\right)\|u^\epsilon(\tau_1)\|_{L^2(K)} \\
& + C\int_0^\tau \left(1+\frac{1}{\sqrt{\tau-\sigma}}\right) e^{-(\tau-\sigma)/2} R_1^\epsilon(\theta_\sigma\omega)\|u^\epsilon(\tau_1+\sigma)\|_{H^1(K)}\,d\sigma \\
& + \tfrac{C}{\sqrt{\epsilon}}\int_0^\tau \left(1+\frac{1}{\sqrt{\tau-\sigma}}\right) e^{-(\tau-\sigma)/2}\big[\|\bar\eta^\epsilon(\tau_1+\sigma)\|_{L^\infty(\mathbb{R})}\|u^\epsilon(\tau_1+\sigma)\|_{H^1(K)} \\
& \qquad + \|\bar\eta^\epsilon\|_{H^1(K)}\|u^\epsilon\|_{L^\infty(\mathbb{R})}\big]\,d\sigma\,.
\end{aligned}$$

By the definition of η^ϵ, the tempered property of R_1^ϵ, and the Gronwall lemma for integral forms, there is a tempered random variable R_2^ϵ such that

$$\|u^\epsilon(\tau+\tau_1)\|_{H^1(K)} \le CR_2^\epsilon(\theta_\tau\omega)(1+\tfrac{1}{\sqrt{\tau}})\|u^\epsilon(\tau_1)\|_{L^2(K)} + R_2^\epsilon(\theta_\tau\omega)$$

for any $\tau_1 > 0$ and $0 < \tau < \tau_1$. Then we have uniform $H^1(K)$-estimates for $u^\epsilon(\tau)$ in τ. By the compact embedding $H^1(K) \subset L^2(K)$ and the classical Bogolyubov–Krylov method [1], a stationary measure exists, denoted by $\bar{\mu}^\epsilon$.

Next we show the emergence of the stationary measure $\bar{\mu}^\epsilon$ by showing it is attractive. We follow the discussion for deterministic systems [40, 21] which was also applied to the case of additive noise in a stochastic Burgers' equation [37]. The following lemma is a key step.

Lemma 7. *For any $u_{1,0}, u_{2,0} \in L^2(K)\cap L^\infty(\mathbb{R})$ with*

$$\int_{\mathbb{R}} u_{1,0}(\xi)\,d\xi = \int_{\mathbb{R}} u_{2,0}(\xi)\,d\xi\,.$$

Let $u_1^\epsilon(\tau,\xi)$ and $u_2^\epsilon(\tau,\xi)$ be the solutions to RDE *(11) with initial value $u_{1,0}$ and $u_{2,0}$ respectively. Then the function*

$$\phi^\epsilon(\tau) = \int_{\mathbb{R}} |u_1^\epsilon(\tau,\xi) - u_2^\epsilon(\tau,\xi)|\,d\xi$$

is strictly decreasing almost surely.

Proof. Let $U^\epsilon(\tau,\xi) = u_1^\epsilon(\tau,\xi) - u_2^\epsilon(\tau,\xi)$, then it satisfies the following linear equation

$$U_\tau^\epsilon = U_{\xi\xi} - \tfrac{1}{2}[(u^1+u^2-\xi-\tfrac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon)U]_\xi\,.$$

Notice that for any solution u^ϵ of the SPDE (11), let $v^\epsilon = u_\xi^\epsilon$, then

$$v_\tau^\epsilon = \mathcal{L}v^\epsilon + \tfrac{1}{2}v^\epsilon - (v^\epsilon)^2 - u^\epsilon v_\xi^\epsilon + \tfrac{1}{\sqrt{\epsilon}}(u^\epsilon\bar{\eta}^\epsilon)_{\xi\xi}\,.$$

By the same discussion as in section 3.1, and the construction of $\bar{\eta}^\epsilon$,

$$(u^1(\tau,\xi) + u^2(\tau,\xi) - \xi - \tfrac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon(\tau,\xi))_\xi$$

is bounded by a random constant for any $\tau > 0$. Then for almost all fixed $\omega \in \Omega_0$, the result follows by the discussion for the corresponding deterministic system [40, 21]. ☐

Now we study the attracting property of any stationary measure. Notice that now u^ϵ is not a Markov process, we consider process $(u^\epsilon, \eta^\epsilon)$ which is a Markov process on space $(L^2(K)\cap L^\infty(\mathbb{R}))\times H^1(K)$.

For this we introduce the space $\mathfrak{M}$ consisting all probability measures on the space $(L^2(K)\cap L^\infty(\mathbb{R}))\times H^1(K)$ and endow the space $\mathfrak{M}$ with the topology of weak convergence. Define the continuous Markov semigroup $\{\mathcal{P}_\tau^\epsilon\}_{\tau\geq 0}$ associating with $(u^\epsilon,\eta^\epsilon)$ on $\mathfrak{M}$ as

$$\mathcal{P}_\tau^{*\epsilon}\mathfrak{u}(A)=\mathbb{P}\{(u^\epsilon(\tau,\cdot),\eta^\epsilon(\tau))\in A\}\,,\quad \mathfrak{u}\in\mathfrak{M}\,,$$

for any Borel measurable set $A\subset(L^2(K)\cap L^\infty(\mathbb{R}))\times H^1(K)$. Here, the solution to equations (11)–(12), with initial value distributing as $\mathfrak{u}$, is $(u^\epsilon(\tau,\cdot),\eta^\epsilon(\tau))$.

Further, we introduce the following subspace of $\mathfrak{M}$; define

$$\mathfrak{M}_2=\left\{\mu\in\mathfrak{M}:\int_{(L^2(K)\cap L^\infty(\mathbb{R}))\times H^1(K)}(\|u\|_{L^2(K)}^2+\|\eta\|_{H^1(K)}^2)\mu(d(u,\eta))<\infty\right\}.$$

Denote by $\bar{\nu}^\epsilon$ the distribution of $\bar{\eta}^\epsilon$ on $H^1(K)$. Then by the construction of $\bar{\mu}^\epsilon\in\mathcal{M}_2$, we have $\mathcal{P}_\tau^{*\epsilon}(\bar{\mu}^\epsilon,\bar{\nu}^\epsilon)=(\bar{\mu}^\epsilon,\bar{\nu}^\epsilon)$, $\tau\geq 0$, which is a stationary measure of $\mathcal{P}_\tau^\epsilon$. Notice that $\bar{\eta}^\epsilon$, which is exponentially stable, is the unique stationary solution to equation (12). So we just consider the projection of $\mathcal{P}_\tau^{*\epsilon}$, which is denoted by $P_\tau^{*\epsilon}$, on $L^2(K)\cap L^\infty(\mathbb{R})$. Then $P_\tau^{*\epsilon}\bar{\mu}^\epsilon=\bar{\mu}^\epsilon$. Moreover for any $\mathfrak{u}=(\mu,\nu)\in\mathfrak{M}_2$, there is a stationary measure $\bar{\mathfrak{u}}$ such that $\mathcal{P}_\tau^{*\epsilon}\mathfrak{u}\to\bar{\mathfrak{u}}$ as $\tau\to\infty$ if and only if for any $\mu\in\mathcal{M}_2$, there is a $\bar{\mu}^\epsilon\in\mathcal{M}_2$ such that $P_\tau^{*\epsilon}\mu\to\bar{\mu}^\epsilon$ as $\tau\to\infty$.

We next show that for any $\mu\in\mathcal{M}_2$ there is a unique stationary measure $\bar{\mu}^\epsilon\in\mathcal{M}_2$ such that $P_\tau^{*\epsilon}\mu$ converges weakly to $\bar{\mu}^\epsilon$ as $\tau\to\infty$.

Associated with the solution to the RDE (11) we choose $\mu\in\mathcal{M}_2$ which has the form

$$\mu=\delta_M*\mu_s\,, \tag{19}$$

where δ_M is some Dirac measure on E_c and μ_s is supported on E_s. Then consider the limit of $P_\tau^{*\epsilon}\mu$ as $\tau\to\infty$. First by the uniform estimates in τ for u^ϵ in $H^1(K)$ we have a measure $\bar{\mu}^\epsilon$, and subsequence τ_n with $\tau_n\to\infty$, $n\to\infty$, such that

$$P_{\tau_n}^{*\epsilon}\mu\to\bar{\mu}^\epsilon\,,\quad n\to\infty\,. \tag{20}$$

We next show $\bar{\mu}^\epsilon$ is unique by Lemma 7. Suppose $\bar{\mu}'^\epsilon$ is another stationary measure such that for some $\tau_n'\to\infty$, $n\to\infty$,

$$P_{\tau_n'}^{*\epsilon}\mu\to\bar{\mu}'^\epsilon\,,\quad n\to\infty\,. \tag{21}$$

Denote by $\bar{u}^\epsilon(\tau,\xi)$ and $\bar{u}'^\epsilon(\tau,\xi)$ the two solutions of RDE (11) with initial value $\bar{u}^1(\xi)$ and $\bar{u}^2(\xi)$, distributed as $\bar{\mu}^\epsilon$ and $\bar{\mu}'^\epsilon$ respectively. Then

$$\int_{\mathbb{R}}\bar{u}^1(\xi)\,d\xi=\int_{\mathbb{R}}\bar{u}^2(\xi)\,d\xi\,.$$

By Lemma 7, the function

$$\int_{\mathbb{R}} |\bar{u}^\epsilon(\tau,\xi) - \bar{u}'^\epsilon(\tau,\xi)|\, d\xi$$

is almost surely strictly decreasing in τ which contradicts the stationarity of $\bar{u}^\epsilon$ and $\bar{u}'^\epsilon$. Hence we deduce the following theorem.

Theorem 8. *Assume Assumption 2 holds. For any initial $u_0 \in L^2(K) \cap L^\infty(\mathbb{R})$, the solution $u^\epsilon(\tau,\xi)$ to* RDE *(11), with initial value u_0, converges in distribution, as $\tau \to \infty$, to $\bar{u}^\epsilon$ in the space $L^2(K)$ which is the unique stationary solution to* SPDE *(11) with*

$$\int_{\mathbb{R}} \bar{u}^\epsilon(\tau,\xi)\, d\xi = \int_{\mathbb{R}} u_0(\xi)\, d\xi\,.$$

Remark 4. By the construction of $\bar{u}^\epsilon$,

$$\|\bar{u}^\epsilon(\tau)\|_{L^\infty(\mathbb{R})} \le m\,, \quad \mathbb{E}\|\bar{u}^\epsilon(\tau)\|^2_{H^1(K)} \le C \quad \text{for all } \tau \ge 0\,, \tag{22}$$

for some constant $C > 0$.

We want to pass the above convergence property to the stochastic Burgers' equation (3), that is we want to pass to the limit $\epsilon \to 0$ in u^ϵ in the space $C([0,\infty), L^2(K))$. We give some estimates uniform in ϵ in the next section.

5 Some a priori estimates on finite time intervals

This section shows the tightness of $\{u^\epsilon\}_{0<\epsilon\le 1}$ in the space $C(0,T;L^2(K))$ for any $T > 0$.

From equation (18), by the chain rule,

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}\|u^\epsilon_s(\tau)\|^2_{L^2(K)} \;&\le\; -\tfrac{1}{2}\|u^\epsilon(\tau)\|^2_{H^1(K)} + \left\langle u^\epsilon(\tau)u^\epsilon_\xi(\tau), u^\epsilon(\tau)\right\rangle \\ &\quad + \left\langle \tfrac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon(\tau)u^\epsilon_\xi(\tau) + \tfrac{1}{\sqrt{\epsilon}}\bar{\eta}^\epsilon_\xi(\tau)u^\epsilon(\tau), u^\epsilon(\tau)\right\rangle . \end{aligned} \tag{23}$$

Define the two integrals

$$\begin{aligned} I^\epsilon_1(\tau) &= \frac{1}{\sqrt{\epsilon}}\int_0^\tau \langle u^\epsilon_\xi(s)\bar{\eta}^\epsilon(s), u^\epsilon(s)\rangle\, ds\,, \\ I^\epsilon_2(\tau) &= \frac{1}{\sqrt{\epsilon}}\int_0^\tau \langle u^\epsilon(s)\bar{\eta}^\epsilon_\xi(s), u^\epsilon(s)\rangle\, ds\,. \end{aligned}$$

Now by the factorization method, for some $0 < \alpha < 1$,

$$\frac{1}{\sqrt{\epsilon}} \int_0^\tau \langle u^\epsilon_\xi(s)\bar{\eta}^\epsilon(s), u^\epsilon(s)\rangle \, ds = \frac{\sin \pi\alpha}{\alpha} \int_0^\tau (\tau - s)^{\alpha-1} Y^\epsilon(s) \, ds$$

where

$$\begin{aligned}
Y^\epsilon(s) &= \frac{1}{\sqrt{\epsilon}} \int_0^s (s-r)^{-\alpha} \langle u^\epsilon_\xi(r)\bar{\eta}^\epsilon(r), u^\epsilon(r)\rangle \, dr \\
&= \frac{1}{2\sqrt{\epsilon}} \int_0^s (s-r)^{\alpha} \langle (u^\epsilon(r))^2_\xi, \bar{\eta}^\epsilon(r)\rangle \, dr \\
&= -\frac{1}{2\sqrt{\epsilon}} \int_0^s (s-r)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(r))^2 \bar{\eta}^\epsilon(r) K_\xi \, d\xi \, dr \\
&\quad - \frac{1}{2\sqrt{\epsilon}} \int_0^s (s-r)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(r))^2 \bar{\eta}^\epsilon_\xi(r) K \, d\xi \, dr \\
&= Y^\epsilon_1(s) + Y^\epsilon_2(s).
\end{aligned}$$

Then for any $T > 0$, there is some positive constant C_T such that

$$\sup_{0 \le \tau \le T} |I^\epsilon_1(\tau)|^2 \le C_T \int_0^T |Y^\epsilon_1(s)|^2 \, ds + C_T \int_0^T |Y^\epsilon_2(s)|^2 \, ds \, .$$

We first consider Y^ϵ_1. By the $L^\infty(\mathbb{R})$ estimates on u^ϵ, and the construction of $\bar{\eta}^\epsilon$,

$$\begin{aligned}
&\mathbb{E}|Y^\epsilon_1(s)|^2 \\
= &\ \frac{1}{\epsilon} \Big| \mathbb{E} \int_0^s \int_\rho^s (s-r)^{-\alpha}(s-\rho)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(r,\xi))^2 \bar{\eta}^\epsilon(r,\xi) K_\xi \, d\xi \\
&\ \times \int_{\mathbb{R}} (u^\epsilon(\rho,\zeta))^2 \bar{\eta}^\epsilon(\rho,\zeta) K_\zeta \, d\zeta \, dr \, d\rho \Big| \\
\le &\ \frac{m^4}{\epsilon} \left| \int_0^s \int_\rho^s (s-r)^{-\alpha}(s-\rho)^{-\alpha} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}\bar{\eta}^\epsilon(r,\xi)\bar{\eta}^\epsilon(\rho,\zeta) K_\xi K_\zeta \, d\xi \, d\zeta \, dr \, d\rho \right| \\
\le &\ C_{1,T} \, ,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{E}|Y^\epsilon_2(s)|^2 \\
= &\ \frac{1}{\epsilon} \Big| \mathbb{E} \int_0^s \int_\rho^s (s-r)^{-\alpha}(s-\rho)^{-\alpha} \int_{\mathbb{R}} (u^\epsilon(r,\xi))^2 \bar{\eta}^\epsilon_\xi(r,\xi) K \, d\xi \\
&\ \times \int_{\mathbb{R}} (u^\epsilon(\rho,\zeta))^2 \bar{\eta}^\epsilon_\zeta(\rho,\zeta) K \, d\zeta \, dr \, d\rho \Big| \\
\le &\ C_{1,T} \, ,
\end{aligned}$$

for some positive constant $C_{1,T}$. Then

$$\mathbb{E}\sup_{0\le\tau\le T}|I_1^\epsilon(\tau)|\le C_T C_{1,T}\,.$$

By the same discussion for I_2^ϵ, a similar expectation holds:

$$\mathbb{E}\sup_{0\le\tau\le T}|I_2^\epsilon(\tau)|\le C_T C_{2,T}\,,$$

for some positive constant $C_{2,T}$. Then by the same discussion for (10) and the Gronwall lemma,

$$\mathbb{E}\sup_{0\le\tau\le T}\|u^\epsilon(\tau)\|^2_{L^2(K)}+\mathbb{E}\int_0^T\|u^\epsilon(s)\|^2_{H^1(K)}\,ds\le C_T \tag{24}$$

for some positive constant C_T. Notice that in the mild sense

$$\begin{aligned} u^\epsilon(\tau) &= S(\tau)u_0+\int_0^\tau S(\tau-\sigma)u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)\,d\sigma\\ &\quad+\frac{1}{\sqrt{\epsilon}}\int_0^\tau S(\tau-\sigma)(u^\epsilon(\sigma)\bar\eta^\epsilon(\sigma))_\xi\,d\sigma\,. \end{aligned}$$

Then for any $T>\tau>\delta>0$,

$$\begin{aligned} &\|u^\epsilon(\tau)-u^\epsilon(\delta)\|_{L^2(K)} \qquad\qquad (25)\\ \le\ &\|(S(\tau)-S(\delta))u_0\|_{L^2(K)}+\left\|\int_\delta^\tau S(\tau-\sigma)u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)\,d\sigma\right\|_{L^2(K)}\\ &+\frac{1}{\sqrt{\epsilon}}\left\|\int_\delta^\tau S(\tau-\sigma)(u^\epsilon(\sigma)\bar\eta^\epsilon(\sigma))_\xi\,d\sigma\right\|_{L^2(K)}\\ &+\left\|\int_0^\delta[S(\tau-\sigma)-S(\delta-\sigma)]u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)\,d\sigma\right\|_{L^2(K)}\\ &+\frac{1}{\sqrt{\epsilon}}\left\|\int_0^\delta[S(\tau-\sigma)-S(\delta-\sigma)](u^\epsilon(\sigma)\bar\eta^\epsilon(\sigma))_\xi\,d\sigma\right\|_{L^2(K)}. \end{aligned}$$

By the $L^\infty(\mathbb{R})$ estimate of u^ϵ, and by estimate (24), the expectation

$$\begin{aligned} &\mathbb{E}\left\|\int_\delta^\tau S(\tau-\sigma)u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)d\sigma\right\|_{L^2(K)}\\ \le\ &\mathbb{E}\int_\delta^\tau\|S(\tau-\sigma)u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)\|_{L^2(K)}d\sigma\\ \le\ &\mathbb{E}\int_\delta^\tau\|u^\epsilon(\sigma)u^\epsilon_\xi(\sigma)\|_{L^2(K)}d\sigma\\ \le\ &m\mathbb{E}\int_\delta^\tau\|u^\epsilon_\xi(\sigma)\|_{L^2(K)}\,d\sigma\le mC_T\sqrt{\tau-\delta}\,. \end{aligned}$$

Expanding by $\{e_k\}_k$ and by (13),

$$
\begin{aligned}
& \frac{1}{\epsilon}\mathbb{E}\left\|\int_\delta^\tau S(\tau-\sigma)(u^\epsilon(\sigma)\bar{\eta}^\epsilon(\sigma))_\xi \, d\sigma\right\|^2_{L^2(K)} \\
\leq\ & \frac{1}{\epsilon}\mathbb{E}\sum_k \int_\delta^\tau \int_\delta^\tau e^{-\lambda_k(\tau-\sigma)} \int_{\mathbb{R}} u^\epsilon(\sigma,\xi)\bar{\eta}^\epsilon(\sigma,\xi)(e_kK)_\xi \, d\xi \\
& \times e^{-\lambda_k(\tau-\lambda)} \int_{\mathbb{R}} u^\epsilon(\lambda,\zeta)\bar{\eta}^\epsilon(\lambda,\zeta)(e_kK)_\zeta \, d\zeta \, d\sigma \, d\lambda \\
\leq\ & \frac{m^2}{\epsilon}\sum_k \int_\delta^\tau \int_\delta^\tau e^{-\lambda_k(\tau-\sigma)} e^{-\lambda_k(\tau-\lambda)} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\bar{\eta}^\epsilon(\sigma,\xi)\bar{\eta}^\epsilon(\lambda,\zeta) \\
& \times (e_kK)_\xi(e_kK)_\zeta \, d\xi \, d\zeta \, d\sigma \, d\lambda \\
\leq\ & C_T(\tau-\delta),
\end{aligned}
$$

for some positive constant C_T. By the strong continuity of the semigroup $S(\tau)$ and a similar discussion, from (25) the expectation

$$
\mathbb{E}\|u^\epsilon(\tau)-u^\epsilon(\delta)\|_{L^2(K)} \leq C_T\sqrt{\tau-\delta}\,. \tag{26}
$$

Now we need the following lemma [25]. Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are two Banach spaces. Let $T>0$, $1\leq p\leq\infty$, and $\mathcal{B}$ be a compact operator from $\mathcal{X}_1$ to $\mathcal{X}_2$; that is, $\mathcal{B}$ maps bounded sets of $\mathcal{X}_1$ to relatively compact subsets of $\mathcal{X}_2$.

Lemma 9 ([25]). *Let H be a bounded subset of $L^1(0,T;\mathcal{X}_1)$ such that $G=\mathcal{B}H$ is a subset of $L^p(0,T;\mathcal{X}_2)$ bounded in $L^r(0,T;\mathcal{X}_2)$ with $r>1$. If*

$$
\lim_{\sigma\to 0}\|u(\cdot+\sigma)-u(\cdot)\|_{L^p(0,T;\mathcal{X}_2)}=0 \quad \textit{uniformly for } u\in G\,,
$$

then G is relatively compact in $L^p(0,T;\mathcal{X}_2)$ (and in $C(0,T;\mathcal{X}_2)$ if $p=+\infty$).

Let $\mathcal{X}_1=H^1(K)$, $\mathcal{X}_2=L^2(K)$ and $\mathcal{B}$ be the embedding from $\mathcal{X}_1$ to $\mathcal{X}_2$, by Lemma 9, from estimates (24) and (26) we obtain the main theorem of this section

Theorem 10. *Assume Assumption 2 holds. For any $T>0$, the distribution of $\{u^\epsilon\}_{0<\epsilon\leq 1}$ is tight in the space $C(0,T;L^2(K))$.*

6 Diffusion approximation

This section determines the limit of u^ϵ, the solutions of the RDE (11), as $\epsilon\to 0$. We first show the tightness of u^ϵ in the space $C([0,\infty),L^2(K))$ by Theorem 10. Then we determine the limit of u^ϵ in the space $C([0,\infty),L^2(K))$ by a martingale approach.

6.1 Tightness in space $C([0,\infty), L^2(K))$

We need the following result on the tightness of a family processes [11, Theorem 3.9.1].

Lemma 11. *Let $\mathcal{X}$ be a Polish space and let $\{X^\epsilon\}_{0<\epsilon\le 1}$ be a family of processes with sample paths in the space $C([0,\infty),\mathcal{X})$. Suppose that for any $\delta > 0$ and $T > 0$ there exists a compact set $\Gamma_{\delta,T} \subset \mathcal{X}$ such that for all $0 < \epsilon \le 1$*

$$\mathbb{P}\{X^\epsilon(t) \in \Gamma_{\delta,T} \text{ for } 0 \le t \le T\} \ge 1 - \delta\,. \tag{27}$$

Then $\{X^\epsilon\}$ is tight in the space $C([0,\infty),\mathcal{X})$ if and only if $\{F(X^\epsilon)\}_{0<\epsilon\le 1}$ is tight in the space $C([0,\infty),\mathbb{R})$ for any $F \in C_b(\mathcal{X})$ the space consisting all continuous and bounded functions on $\mathcal{X}$.

Remark 5. We do not need to verify the tightness of $\{F(X^\epsilon)\}_{0<\epsilon\le 1}$ for all $F \in C_b(\mathcal{X})$. One just needs to verify the tightness for all F in a dense subset of $C_b(\mathcal{X})$ in the topology of uniform convergence on compact sets [11, Theorem 3.9.1].

By Theorem 10, the pre-condition (27) in Lemma 11 holds. Next we show the tightness of $\{F(u^\epsilon)\}_{0<\epsilon\le 1}$ in the space $C([0,\infty))$ for any $F \in C_b(L^2(K))$. We follow a martingale approach. Continue to let $\mathcal{X}$ be a Polish space and $\{X^\epsilon\}_{0<\epsilon\le 1}$ be a family of processes valued in the space $C([0,\infty),\mathcal{X})$ adapted to the filtration $\mathcal{F}^\epsilon_\tau$. Let $\mathfrak{L}^\epsilon$ be the Banach space of real valued $\mathcal{F}^\epsilon_\tau$-progressive processes with norm $\|Y\|_{\mathfrak{L}^\epsilon} = \sup_{\tau\ge 0}\mathbb{E}|Y(\tau)|$. Let

$$\mathcal{M}^\epsilon = \left\{(Y,Z) \in \mathfrak{L}^\epsilon \times \mathfrak{L}^\epsilon : Y(\tau) - \int_0^\tau Z(s)\,ds \text{ is } \mathcal{F}^\epsilon_\tau\text{-martingale}\right\}. \tag{28}$$

Then the following lemma applies [11, Theorem 3.9.4].

Lemma 12. *For any bounded continuous function F on $\mathcal{X}$ with bounded support, and for any $\delta > 0$ and $T > 0$, there is $(Y^\epsilon, Z^\epsilon) \in \mathcal{M}^\epsilon$ such that*

$$\limsup_{\epsilon\to 0}\mathbb{E}\left[\sup_{\tau\in[0,T]}|Y^\epsilon(\tau) - F(X^\epsilon(\tau))|\right] < \delta \tag{29}$$

and

$$\sup_\epsilon \mathbb{E}\left[\|Z^\epsilon\|_{L^p(0,T)}\right] < \infty \quad \text{for some } p \in (1,\infty]. \tag{30}$$

Then $\{F(X^\epsilon)\}_{0<\epsilon\le 1}$ is tight in $C([0,\infty),\mathbb{R})$.

By Remark 5 we just need to show the tightness of the following family of real valued processes [22]:

$$\{f(\langle u^\epsilon, \varphi\rangle)\}_{0<\epsilon\le 1}$$

for any $\varphi \in \mathcal{D}(\mathbb{R})$ and twice differentiable compactly supported functions f. From the RDE (11)

$$\begin{aligned} & f(\langle u^\epsilon(\tau), \varphi\rangle) - f(\langle u_0, \varphi\rangle) \qquad (31)\\ = & \int_0^\tau f'(\langle u^\epsilon(s), \varphi\rangle)\langle u^\epsilon(s), \mathcal{L}\varphi\rangle ds - \int_0^\tau f'(\langle u^\epsilon(s), \varphi\rangle)\langle u^\epsilon(s) u^\epsilon_\xi(s), \varphi\rangle ds \\ & + \frac{1}{\sqrt{\epsilon}}\int_0^\tau f'(\langle u^\epsilon(s), \varphi\rangle)\langle (u^\epsilon(s)\bar{\eta}^\epsilon(s))_\xi, \varphi\rangle ds\,. \end{aligned}$$

One can see that the singular term in the above equation is difficult to treat. To treat this term we follow a perturbation approach developed by Kushner [22]. Let $\mathcal{F}^\epsilon_\tau$ be the σ-algebra generated by $\{\bar{\eta}^\epsilon(s) : 0 \le s \le \tau\}$. Then introduce the process

$$F^\epsilon_1(\tau) = \frac{1}{\sqrt{\epsilon}}\mathbb{E}\left[\int_\tau^\infty f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(s))_\xi, \varphi\rangle ds \,\middle|\, \mathcal{F}^\epsilon_\tau\right]. \qquad (32)$$

For the process $F^\epsilon_1(\tau)$ the following lemma holds.

Lemma 13. *Assume Assumption 2 holds. Then*

$$F^\epsilon_1(\tau) = \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi, \varphi\rangle\,. \qquad (33)$$

Furthermore,

$$\mathbb{E}|F^\epsilon_1(\tau)| \le C\sqrt{\epsilon}$$

for some positive constant C, and for any $T > 0$

$$\mathbb{E}\sup_{0\le\tau\le T}|F^\epsilon_1(\tau)| \to 0\,, \quad \textit{as } \epsilon \to 0\,.$$

Proof. The equality (33) is implied by (13) and the property of conditional expectation. Then by the L^∞ bound on u^ϵ and the estimates on $\bar{\eta}^\epsilon$,

$$\begin{aligned} \mathbb{E}|F^\epsilon_1(\tau)| & \le \sqrt{\epsilon}\|f'\|_{L^\infty(\mathbb{R})}\|u^\epsilon(\tau)\|_{L^\infty(\mathbb{R})}\mathbb{E}\|\bar{\eta}^\epsilon(\tau)\|_{L^2(K)}\left\|\varphi_\xi + \tfrac{1}{2}\xi\varphi\right\|_{L^2(K)} \\ & \le C\sqrt{\epsilon} \end{aligned}$$

for some positive constant C. Further, by the maximal estimate on stochastic integral [29, Lemma 7.2], for any $T > 0$

$$\mathbb{E}\sup_{0\le\tau\le T}\|\bar{\eta}^\epsilon(\tau)\|^2_{L^2(K)} \le C_T$$

for some positive constant C_T. Then by (33)

$$\mathbb{E}\sup_{0\le\tau\le T}|F^\epsilon_1(\tau)| \to 0\,, \quad \text{as } \epsilon \to 0\,.$$

The proof is complete. ∎

To apply Lemma 12, we first construct $(Y^\epsilon, Z^\epsilon) \in \mathfrak{L}^\epsilon$. For this we introduce the operator A^ϵ defined by

$$A^\epsilon f(\tau) = \mathbb{P} - \lim_{\delta\to 0} \tfrac{1}{\delta} \left[\mathbb{E} f(\tau+\delta) - f(\tau) \mid \mathcal{F}^\epsilon_\tau\right] \tag{34}$$

for any $\mathcal{F}^\epsilon_\tau$ measurable function f with $\sup_\tau \mathbb{E}|f(\tau)| < \infty$. Then Ethier and Kurtz's proposition [11, Proposition 2.7.6] yields that

$$f(\tau) - \int_0^\tau A^\epsilon f(s)\, ds$$

is a martingale with respect to $\mathcal{F}^\epsilon_\tau$. Now define (Y^ϵ, Z^ϵ) as

$$Y^\epsilon(\tau) = f(\langle u^\epsilon(\tau), \varphi\rangle) - F_1^\epsilon(\tau), \quad Z^\epsilon(\tau) = A^\epsilon Y^\epsilon(\tau).$$

Then we establish the following lemma.
Lemma 14.

$$\begin{aligned} Z^\epsilon(\tau) \ &= \ f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle u^\epsilon(\tau), \mathcal{L}\varphi\rangle + f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle \tfrac{1}{2}(u^\epsilon(\tau))^2_\xi, \varphi\rangle \\ &\quad + f''(\langle u^\epsilon(\tau), \varphi\rangle)\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau), \varphi_\xi + \tfrac{1}{2}\xi\varphi_\xi\rangle^2 \\ &\quad + f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi, (\varphi_\xi + \tfrac{1}{2}\xi\varphi_\xi)\bar{\eta}^\epsilon(\tau)\rangle \\ &\quad - \sqrt{\epsilon} f''(\langle u^\epsilon(\tau), \varphi\rangle)[\langle u^\epsilon(\tau), \mathcal{L}\varphi\rangle + \langle \tfrac{1}{2}(u^\epsilon(\tau))^2_\xi, \varphi\rangle] \\ &\quad - \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi\rangle)\left[\langle u^\epsilon(\tau), \mathcal{L}((\varphi_\xi + \tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau))\right. \\ &\qquad \left. + \langle \tfrac{1}{2}(u^\epsilon(\tau))^2_\xi, (\varphi_\xi + \tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau)\rangle\right]. \end{aligned}$$

Proof. By (31),

$$\begin{aligned} A^\epsilon f(\langle u^\epsilon(\tau), \varphi) \ &= \ f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle u^\epsilon(\tau), \mathcal{L}\varphi\rangle \\ &\quad - f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle u^\epsilon(\tau) u^\epsilon_\xi(\tau), \varphi\rangle \\ &\quad + \tfrac{1}{\sqrt{\epsilon}} f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi, \varphi\rangle\,. \end{aligned}$$

Now consider $A^\epsilon F_1^\epsilon$. By (33) and the construction of $\bar{\eta}^\epsilon$,

$$\begin{aligned} &\mathbb{E}[F_1^\epsilon(\tau+\delta) \mid \mathcal{F}^\epsilon_\tau] \\ = \ &\sqrt{\epsilon}\mathbb{E}\big\{[f'(\langle u^\epsilon(\tau+\delta), \varphi\rangle) - f'(\langle u^\epsilon(\tau), \varphi\rangle)] \\ &\times \langle u^\epsilon(\tau+\delta)\bar{\eta}^\epsilon(\tau+\delta), \phi_\xi + \tfrac{1}{2}\xi\varphi\rangle \mid \mathcal{F}^\epsilon_\tau\big\} \\ &+ \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi\rangle)\mathbb{E}[\langle (u^\epsilon(\tau+\delta) - u^\epsilon(\tau))\bar{\eta}^\epsilon(\tau+\delta), \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle \mid \mathcal{F}^\epsilon_\tau] \\ &+ \sqrt{\epsilon} f'(\langle u^\epsilon(\tau), \varphi\rangle)\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau)e^{-\delta/\epsilon}, \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle\,. \end{aligned}$$

Then

$$
\begin{aligned}
A^\epsilon F_1^\epsilon(\tau) \;\; = \;\; & -f''(\langle u^\epsilon(\tau),\varphi\rangle)\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau),\varphi_\xi+\tfrac{1}{2}\xi\varphi_\xi\rangle^2 \\
& +\sqrt{\epsilon}f''(\langle u^\epsilon(\tau),\varphi\rangle)[\langle u^\epsilon(\tau),\mathcal{L}\varphi\rangle+\langle\tfrac{1}{2}(u^\epsilon(\tau))^2_\xi,\varphi\rangle] \\
& +\frac{1}{\sqrt{\epsilon}}f'(\langle u^\epsilon(\tau),\varphi\rangle)\langle(u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi,\varphi\rangle \\
& -f'(\langle u^\epsilon,\varphi\rangle)\langle(u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi,(\varphi_\xi+\tfrac{1}{2}\xi\varphi_\xi)\bar{\eta}^\epsilon(\tau)\rangle \\
& +\sqrt{\epsilon}f'(\langle u^\epsilon(\tau),\varphi\rangle)\left[\langle u^\epsilon(\tau),\mathcal{L}((\varphi_\xi+\tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau))\right. \\
& \left.-\langle\tfrac{1}{2}(u^\epsilon(\tau))^2_\xi,(\varphi_\xi+\tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau)\rangle\right].
\end{aligned}
$$

This completes the proof. □

Now by the above construction of (Y^ϵ, Z^ϵ),

$$Y^\epsilon(\tau)-f(\langle u^\epsilon(\tau),\varphi\rangle)=-F_1^\epsilon(\tau).$$

Then by Lemma 13,

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{0\le\tau\le T}|F_1^\epsilon(\tau)|=0\,.$$

Furthermore, by the $L^\infty(\mathbb{R})$ estimates on $u^\epsilon(\tau)$,

$$\sup_\epsilon\mathbb{E}\|Z^\epsilon(\tau)\|_{L^2(0,T)}<\infty\,. \tag{35}$$

By Lemma 12, we thus deduce the following theorem.

Theorem 15. *Assume Assumption 2 holds. The family of processes* $\{u^\epsilon\}_{0<\epsilon\le 1}$ *is tight in the space* $C([0,\infty),L^2(K))$.

Remark 6. By (22), all discussions in this section and section 5 hold for $\bar{u}^\epsilon$. So the above tightness result also holds for $\{\bar{u}^\epsilon\}_{0<\epsilon\le 1}$.

6.2 Pass to the limit $\epsilon\to 0$

We show the convergence of u^ϵ as $\epsilon\to 0$ and determine the limit. For this we introduce a limit martingale problem and show any accumulation point of $\{u^\epsilon\}$ is a solution to this martingale problem. By the convergence result of Walsh [36, Theorem 6.15], we just need to consider finite dimensional distributions of $\{\langle u^\epsilon,\varphi_1\rangle,\dots,\langle u^\epsilon,\varphi_n\rangle\}$ for any $\varphi_1,\dots,\varphi_n\in\mathcal{D}(\mathbb{R})$; that is, we just pass to the limit $\epsilon\to 0$ in the following equality

$$\mathbb{E}\left\{\left[Y^\epsilon(\tau)-Y^\epsilon(s)-\int_s^\tau Z^\epsilon(r)\,dr\right]h(\langle u^\epsilon(r_1),\varphi_1\rangle,\dots,\langle u^\epsilon(r_n),\varphi_n\rangle)\right\}=0 \tag{36}$$

for any bounded continuous function h and $0 < r_1 < \cdots < r_n < T$ with any $T > 0$. Denote by u one limit point in the sense of distribution of u^ϵ as $\epsilon \to 0$ in space $D([0,\infty), L^2(K))$. By the estimates in Lemma 13 and the construction of Y^ϵ, the limit of $Y^\epsilon(\tau) - Y^\epsilon(s)$ is $f(\langle u(\tau), \varphi\rangle) - f(\langle u(s), \varphi\rangle)$.

Now we treat the integral term. First denote by $Z_k^\epsilon(\cdot)$, $k = 1, 2, 3, 4$, the first four terms of $Z^\epsilon(\cdot)$ and by $Z_5^\epsilon(\cdot)$ the last two terms in $Z^\epsilon(\cdot)$. Then, in distribution as $\epsilon \to 0$,

$$\int_s^\tau Z_1^\epsilon(r)\, dr \to \int_s^\tau f'(\langle u(r), \varphi\rangle)\langle u(r), \mathcal{L}\varphi\rangle\, dr\,,$$

and by the estimates on u^ϵ in section 5 and estimates on $\bar{\eta}^\epsilon$ in section 2,

$$\mathbb{E}\int_s^\tau |Z_5^\epsilon(r)| dr \to 0\,.$$

Notice that $(u^\epsilon)^2$ is bounded in $L^2(0, T; L^2(K))$ for any $T > 0$ and by the tightness u^ϵ in the space $C(0, T; L^2(K))$, u^ϵ converges almost everywhere to u on $[0, T] \times \mathbb{R}$, then by the $L^\infty(\mathbb{R})$ bound on u^ϵ and u, we have in distribution for any $T > 0$

$$\langle (u^\epsilon)^2, \varphi\rangle \to \langle u^2, \varphi\rangle \quad \text{as } \epsilon \to 0\,. \tag{37}$$

So in distribution as $\epsilon \to 0$

$$\int_s^\tau Z_2^\epsilon(r) dr \to \int_s^\tau f'(\langle u(r), \varphi\rangle)\langle u(r)u_\xi(r), \varphi\rangle dr.$$

Next we treat terms Z_3^ϵ and Z_4^ϵ. For any $u \in L^2(K)$ define the bilinear operator $\Sigma(u)$ such that

$$\langle \Sigma(u)\varphi, \varphi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} u(\xi)u(\zeta)q(\xi,\zeta)(\varphi(\xi)K(\xi))_\xi(\varphi(\zeta)K(\zeta))_\zeta\, d\xi\, d\zeta \tag{38}$$

for any $\varphi \in \mathcal{D}(\mathbb{R})$, and define the linear operator

$$\begin{aligned}\langle A(u), \varphi\rangle &= \tfrac{1}{2}\int_{\mathbb{R}} u(\xi)q(\xi,\xi)(\varphi(\xi)K(\xi))_{\xi\xi}\, d\xi \\ &\quad + \tfrac{1}{2}\int_{\mathbb{R}} u(\xi)q'(\xi,\xi)(\varphi(\xi)K(\xi))_\xi\, d\xi\,.\end{aligned} \tag{39}$$

For this operator Σ the following lemma holds.

Lemma 16. *For any $u \in H^1(K)$, let $B(\tau,\xi) = (u(\xi)W(\tau,\xi))_\xi$, then B is an $L^2(K)$ valued Wiener process with the covariation operator $\Sigma(u)$.*

Proof. The proof is direct. By (6),

$$W_\xi = \sum_{k=1}^{\infty} \sqrt{q_k} e_k'(\xi) \beta_k(\tau).$$

Then by the representation of $q(\xi, \zeta)$,

$$\begin{aligned}
& \mathbb{E}B(\tau,\xi)B(\tau,\zeta) \\
= \; & \mathbb{E}\left(u(\xi)\sum_k \sqrt{q_k} e_k(\xi)\beta_k(\tau)\right)_\xi \left(u(\zeta)\sum_l \sqrt{q_l} e_l(\zeta)\beta_l(\tau)\right)_\xi \\
= \; & u_\xi(\xi)u(\zeta)q_\zeta(\xi,\zeta) + u_\xi(\xi)u_\zeta(\zeta)q(\xi,\zeta) + u(\xi)u_\zeta(\zeta)q_\xi(\xi,\zeta) \\
& + u(\xi)u(\zeta)q_{\xi,\zeta}(\xi,\zeta).
\end{aligned}$$

By the definition of $\Sigma(u)$ this proves the lemma. $\square$

To pass to the limit $\epsilon \to 0$ in Z_3^ϵ and Z_4^ϵ we apply again the perturbation method [22]. Let

$$\begin{aligned}
F_3^\epsilon(\tau) \;\; = \;\; & f''(\langle u^\epsilon(\tau), \varphi\rangle) \int_\tau^\infty \mathbb{E}\Big[\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(s), \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle^2 \\
& - \langle \Sigma(u^\epsilon(\tau))\varphi, \varphi\rangle \,\Big|\, \mathcal{F}_\tau^\epsilon\Big]\, ds
\end{aligned}$$

and

$$\begin{aligned}
F_4^\epsilon(\tau) \;\; = \;\; & f'(\langle u^\epsilon(\tau), \varphi\rangle) \int_\tau^\infty \mathbb{E}\Big[\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(s))_\xi, (\varphi_\xi + \tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(s)\rangle \\
& - \langle A(u^\epsilon(\tau)), \varphi\rangle \,\Big|\, \mathcal{F}_\tau^\epsilon\Big] ds\,.
\end{aligned}$$

By the construction of $\bar{\eta}^\epsilon$ and $\bar{\eta}_\xi^\epsilon$,

$$\mathbb{E}\left[\bar{\eta}^\epsilon(s,\xi)\bar{\eta}^\epsilon(s,\zeta) \mid \mathcal{F}_\tau^\epsilon\right] = e^{-2(s-\tau)/\epsilon}\bar{\eta}^\epsilon(\tau,\xi)\bar{\eta}^\epsilon(\tau,\zeta) + \tfrac{1}{2}q(\xi,\zeta)(1 - e^{-2(s-\tau)/\epsilon}), \tag{40}$$

$$\mathbb{E}\left[\bar{\eta}^\epsilon(s,\xi)\bar{\eta}_\xi^\epsilon(s,\xi) \mid \mathcal{F}_\tau^\epsilon\right] = e^{-2(s-\tau)/\epsilon}\bar{\eta}^\epsilon(\tau,\xi)\bar{\eta}_\xi^\epsilon(\tau,\xi) + \tfrac{1}{2}q'(\xi,\xi)(1 - e^{-2(s-\tau)/\epsilon}). \tag{41}$$

Then

$$F_3^\epsilon(\tau) = \tfrac{\epsilon}{2} f''(\langle u^\epsilon(\tau), \varphi\rangle)\big[\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau), \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle^2 - \langle \Sigma(u^\epsilon(\tau))\varphi, \varphi\rangle\big] \tag{42}$$

and

$$F_4^\epsilon(\tau) = \tfrac{\epsilon}{2} f'(\langle u^\epsilon(\tau), \varphi\rangle)\big[\langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi, (\varphi_\xi + \tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau)\rangle - \langle A(u^\epsilon(\tau)), \varphi\rangle\big]\,. \tag{43}$$

Then by the estimates on $u^\epsilon(\tau)$, $\bar{\eta}^\epsilon(\tau)$ and $\bar{\eta}_\xi^\epsilon(\tau)$, direct computation yields this lemma.

Lemma 17. *As* $\epsilon \to 0$ *,*

$$\sup_{\tau\geq 0} \mathbb{E}F_3^\epsilon(\tau) = \mathcal{O}(\epsilon), \quad \textit{and} \quad \sup_{\tau\geq 0} \mathbb{E}F_4^\epsilon(\tau) = \mathcal{O}(\epsilon).$$

Now following the same discussion as in Lemma 14 and (40)–(41), we have the following lemma.

Lemma 18.

$$\begin{aligned} A^\epsilon F_3^\epsilon(\tau) &= f''(\langle u^\epsilon(\tau), \varphi)[\langle \Sigma(u^\epsilon(\tau))\varphi, \varphi\rangle \\ &\quad - \langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau), \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle] + R_3^\epsilon(\tau) \end{aligned}$$

and

$$\begin{aligned} A^\epsilon F_4^\epsilon(\tau) &= f'(\langle u^\epsilon(\tau), \varphi\rangle)[\langle Au^\epsilon(\tau), \varphi\rangle \\ &\quad - \langle (u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau))_\xi, (\varphi_\xi + \tfrac{1}{2}\xi\varphi)\bar{\eta}^\epsilon(\tau)] + R_4^\epsilon(\tau) \end{aligned}$$

with

$$\sup_{\tau\geq 0} \mathbb{E}|R_3^\epsilon(\tau)| = \mathcal{O}(\epsilon) \quad \textit{and} \quad \sup_{\tau\geq 0} \mathbb{E}|R_4^\epsilon(\tau)| = \mathcal{O}(\epsilon)$$

as $\epsilon \to 0$ *.*

Proof. This is similar to the discussion in the proof of Lemma 14. First we have for any $\delta > 0$

$$\begin{aligned} &\mathbb{E}[F_3^\epsilon(\tau+\delta) \mid \mathcal{F}_\tau^\epsilon] \\ = \ &\tfrac{\epsilon}{2}\mathbb{E}\Big[\big(f''(\langle u^\epsilon(\tau+\delta), \varphi\rangle) - f''(\langle u^\epsilon(\tau), \varphi\rangle)\big) \\ &\times \big(\langle u^\epsilon(\tau+\delta)\bar{\eta}^\epsilon(\tau+\delta), \varphi_\xi + \tfrac{1}{2}\xi\varphi\rangle^2 \\ &\quad - \langle \Sigma(u^\epsilon(\tau+\delta))\varphi, \varphi\rangle\big) \mid \mathcal{F}_\tau^\epsilon\Big] \\ &+ \tfrac{\epsilon}{2} f''(\langle u^\epsilon(\tau), \varphi\rangle)\mathbb{E}\Big[\langle (u^\epsilon(\tau+\delta) - u^\epsilon(\tau))\bar{\eta}^\epsilon(\tau+\delta), \varphi\rangle^2 \\ &\quad - \langle (\Sigma(u^\epsilon(\tau+\delta)) - \Sigma(u^\epsilon(\tau)))\varphi, \varphi\rangle \mid \mathcal{F}_\tau^\epsilon\Big] \\ &+ \tfrac{\epsilon}{2} f''(\langle u^\epsilon(\tau), \varphi\rangle)\mathbb{E}\Big[\langle u^\epsilon(\tau)\bar{\eta}^\epsilon(\tau+\delta), \varphi\rangle^2 - \langle \Sigma(u^\epsilon(\tau))\varphi, \varphi\rangle \mid \mathcal{F}_\tau^\epsilon\Big] \end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}[F_4^\epsilon(\tau+\delta) \mid \mathcal{F}_\tau^\epsilon] \\
= \;& \tfrac{\epsilon}{2}\mathbb{E}\Big[\big(f'(\langle u^\epsilon(\tau+\delta),\varphi\rangle) - f'(\langle u^\epsilon(\tau),\varphi\rangle)\big) \\
&\times \big(\langle (u^\epsilon(\tau+\delta)\bar\eta^\epsilon(\tau+\delta))_\xi, (\varphi_\xi + \tfrac12\xi\varphi)\bar\eta^\epsilon(\tau+\delta)\rangle \\
&\quad -\langle A(u^\epsilon(\tau+\delta)),\varphi\rangle\big) \mid \mathcal{F}_\tau^\epsilon\Big] \\
&+ \tfrac{\epsilon}{2} f'(\langle u^\epsilon(\tau),\varphi\rangle)\mathbb{E}\Big[\big\langle [(u^\epsilon(\tau+\delta) - u^\epsilon(\tau))\bar\eta^\epsilon(\tau+\delta)]_\xi, \\
&\quad (\varphi_\xi + \tfrac12\xi\varphi_\xi)\bar\eta^\epsilon(\tau+\delta)\big\rangle - \langle A(u^\epsilon(\tau+\delta)) - A(u^\epsilon(\tau)),\varphi\rangle \mid \mathcal{F}_\tau^\epsilon\Big] \\
&+ \tfrac{\epsilon}{2} f'(\langle u^\epsilon(\tau),\varphi\rangle)\mathbb{E}\Big[\langle (u^\epsilon(\tau)\bar\eta^\epsilon(\tau+\delta))_\xi, (\varphi_\xi + \tfrac12\xi\varphi)\bar\eta^\epsilon(\tau+\delta)\rangle \\
&\quad -\langle A(u^\epsilon(\tau)),\varphi\rangle \mid \mathcal{F}_\tau^\epsilon\Big].
\end{aligned}
$$

Then by the definition of A^ϵ and (40)–(41), direct computation yields the result. The proof is complete. □

Now we have the following $\mathcal{F}_\tau^\epsilon$ martingale

$$
\begin{aligned}
\mathcal{M}^\epsilon(\tau) \;=\;& f(\langle u^\epsilon(\tau),\varphi\rangle) - F_1^\epsilon(\tau) - F_3^\epsilon(\tau) - F_4^\epsilon(\tau) \\
&- \int_0^\tau f'(\langle u^\epsilon(\tau),\varphi\rangle)\Big[\langle u^\epsilon(r),\mathcal{L}\varphi\rangle + \big\langle \tfrac12 (u^\epsilon(r))^2, \varphi_\xi + \tfrac12\xi\varphi\big\rangle \\
&\qquad\qquad + \langle A(u^\epsilon(r)),\varphi\rangle\Big]dr \\
&- \tfrac12 \int_0^\tau f''(\langle u^\epsilon(r),\varphi\rangle)\langle \Sigma(u^\epsilon(r))\varphi,\varphi\rangle dr + R^\epsilon(\tau)
\end{aligned}
$$

where

$$R^\epsilon(\tau) = \int_0^\tau [Z_5^\epsilon(s) + R_3^\epsilon(s) + R_4^\epsilon(s)]\,ds$$

with $\mathbb{E}|R^\epsilon(\tau)| = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Now passing to the limit $\epsilon \to 0$, the distribution of the limit u solves the martingale problem

$$
\begin{aligned}
\mathcal{M}(\tau) \;=\;& f(\langle u(\tau),\varphi\rangle) - \int_0^\tau f'(\langle u(\tau),\varphi\rangle) \\
&\times \Big[\langle u(r),\mathcal{L}\varphi\rangle + \big\langle \tfrac12 (u(r))^2, \varphi_\xi + \tfrac12\xi\varphi\big\rangle + \langle A(u(r)),\varphi\rangle\Big]dr \\
&- \tfrac12 \int_0^\tau f''(\langle u(r),\varphi\rangle)\langle \Sigma(u(r))\varphi,\varphi\rangle dr
\end{aligned}
\tag{44}
$$

which, by Lemma 16, is equivalent to the martingale solution to the SPDE [27]

$$u_\tau = \mathcal{L}u - uu_\xi + \tfrac12 (uq)_{\xi\xi} - \tfrac12 (uq')_\xi + (u\dot{\bar{W}})_\xi \tag{45}$$

for some new Wiener process $\bar{W}$ with the same distribution as that of W. By the general theory of SPDEs [29], the martingale solution to the SPDE (45) is unique in the space $L^2(0,T;H^1(K))\cap C(0,T;L^2(K))$ for any $T>0$. Then we deduce the following theorem.

Theorem 19. *Assume Assumption 2 holds. The solution of* RDE *(11), u^ϵ, converges in distribution in the space $C([0,\infty),L^2(K))$ to u which solves the* SPDE *(45).*

Notice that the SPDE (45) is different from the stochastic Burgers' equation (3). For this we consider the following RDE

$$u^\epsilon_\tau=\mathcal{L}u^\epsilon-u^\epsilon u^\epsilon_\xi-\tfrac{1}{2}(u^\epsilon q)_{\xi\xi}+\tfrac{1}{2}(u^\epsilon q')_\xi+\tfrac{1}{\sqrt{\epsilon}}(u^\epsilon\bar{\eta}^\epsilon)_\xi\,. \tag{46}$$

By the assumption that $q(\xi)\in H^2(K)$ and (5), the extra terms $\frac{1}{2}(u^\epsilon q)_{\xi\xi}$ and $\frac{1}{2}(u^\epsilon q')_\xi$ do not change the estimates in sections 3–6. Then we derive this theorem.

Theorem 20. *Assume Assumption 2 holds. The solution of the* RDE *(46) converges in distribution in space $C([0,\infty),L^2(K))$ to a solution of the* SPDE *(3).*

Assume u^ϵ is a solution of (46) with initial value $u_0\in L^2(K)\cap L^\infty(\mathbb{R})$ and $u^{*\epsilon}$ is unique the stationary solution such that

$$u^\epsilon(\tau)\text{ converges in distribution to }\bar{u}^\epsilon(\tau)\text{ in }L^2(K)\text{ as }\tau\to\infty\,.$$

Now assume $\bar{u}$ is the limit of $\bar{u}^\epsilon$ in distribution. Then for the solution u of the SPDE (3) with initial value u_0

$$u(\tau)\text{ converges in distribution to }\bar{u}(\tau)\text{ in }L^2(K)\text{ as }\tau\to\infty\,. \tag{47}$$

Then we draw the following main theorem.

Theorem 21. *Assume Assumption 2 holds. For any solution of the stochastic Burgers' equation (2) with $w(1,x)\in L^2(K)\cap L^\infty(\mathbb{R})$, there is a unique self-similar solution $\bar{w}(t,x)$ such that*

$$\sqrt{t}w(t,x)-\sqrt{t}\bar{w}(t,x)\to 0\,,\quad \textit{as } t\to\infty\quad \textit{in distribution in } L^2(\mathbb{R}).$$

Acknowledgements This research is supported by the Australian Research Council grants DP0988738.

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